Exercises

Fourier Series

The exercises are split into the following three categories:

- The exercises in the table below are Mandatory. These exercises must be prepared in such a way that they can be presented during the compulsory practice hours.
- Pencast [P] exercises, from which a complete work-out is available in a pencast video.
- The resulting exercises are available for additional training.

Subject		Exercise
Fourier Series	M1	Ex.4
	M2	Ex.5
	M3	Ex.7
	M4	Ex.9

Exercise 1
Fig. 1 is the spectral plot of signal $x(t)$.

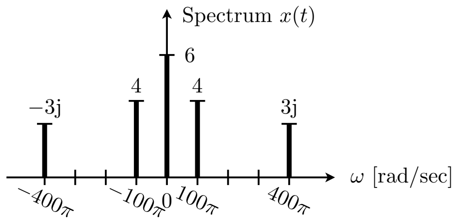

Figure 1: Spectrum of $x(t)$.

a. Write an equation for $x(t)$ in terms of sinusoidal signals:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k).$$

b. Determine the fundamental period T_0 of $x(t)$.

c. Write this signal as a Fourier series of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$$

in which F_0 denotes the fundamental frequency $F_0 = 1/T_0$. Determine which coefficients α_k (spectral weights) have non-zero value. List these Fourier series coefficients and their values.

Exercise 2
A periodic signal $x(t)$ is given by

$$x(t) = 1 + 3\cos(300\pi t) + 2\sin(500\pi t - \pi/4)$$

a. This signal is a periodic signal. Thus we can write it as a Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$, with the fundamental frequency $F_0 = 1/T_0$. What is the fundamental period T_0 of $x(t)$?

b. Find the Fourier series coefficients α_k of $x(t)$.

Exercise 3
The frequency spectrum of the signal $x(t)$ is shown in Fig. **??**. This signal is a periodic signal. Thus we can write it as a Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\omega_0 kt}$, with the fundamental frequency ω_0. Determine ω_0 as well as the Fourier coefficients α_k of $x(t)$.

Exercise 4
An amplitude-modulated signal $x(t)$ can be written as

$$x(t) = s(t) \cdot g(t).$$

The carrier signal $g(t)$, with carrier frequency $f_c = 10000$ [Hz], and the message signal $s(t)$ are given as

$$g(t) = \cos(2\pi f_c t) \qquad \text{and} \qquad s(t) = 1 + \cos(500\pi t + \pi/2)$$

a. Draw the frequency spectrum of $x(t)$, with on the horizontal axis the frequency f in [Hz].

b. Since $x(t)$ is periodic we are able to write it as a Fourier series $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$ with the fundamental frequency $F_0 = 1/T_0$ and the Fourier coefficients α_k. Evaluate F_0 and the coefficients α_k.

Exercise 5

A signal composed of sinusoidal signals is given by the equation:

$$x(t) = 3\cos(50\pi t - \pi/8) - 5\cos(150\pi t + \pi/6)$$

a. Is $x(t)$ periodic? If so, what is the fundamental period $T_{0,x}$? Which harmonics are present?

b. Now consider a new signal:

$$y(t) = x(t) + 7\cos(160\pi t - \pi/3).$$

How is the spectrum changed? Is $y(t)$ periodic? If so, what is the fundamental period $T_{0,y}$?

c. Finally, consider another new signal

$$w(t) = x(t) + \cos\left(5\sqrt{2}\pi t + \pi/3\right).$$

How is the spectrum changed? Is $w(t)$ periodic? If so, what is the fundamental period $T_{0,w}$? If not, why not?

Exercise 6

[P3] A periodic signal $x(t)$ with a period $T_0 = 4$ is described *over one period*, $0 \leq t \leq T_0$, by the equation

$$x(t) = \begin{cases} 2 & 0 \leq t \leq 2 \\ 0 & 2 < t \leq 4 \end{cases}$$

a. Sketch the periodic function $x(t)$ for $-4 < t < 8$.

b. Determine the DC coefficient α_0 of the Fourier Series.

c. Use the Fourier analysis integral (for $k \neq 0$)

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi F_0 kt} dt \quad \text{with fundamental frequency } F_0 = 1/T_0$$

to find the Fourier series coefficients, α_k.

d. This periodic signal $x(t)$ can be expressed with the Fourier series as:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$$

In practice we approximate such a periodic signal with a finite number of harmonics as follows:

$$\hat{x}(t) = \sum_{k=-N}^{N} \alpha_k e^{j2\pi F_0 kt}$$

Make a sketch of $\hat{x}(t)$ for $N = 1$ showing that this approximation with one harmonic is a reasonable approximation of $x(t)$.

e. Now we replace this periodic signal $x(t)$ with another related periodic signal $y(t)$ which is defined as:

$$y(t) = 2x(t + \frac{T_0}{2}) - 1$$

Since $y(t)$ is again periodic with the same period T_0 we can write it as the following Fourier series:

$$y(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{j2\pi F_0 kt}$$

How are the Fourier coefficients β_k of signal $y(t)$ related to the Fourier coefficients α_k of signal $x(t)$? Try to give a physical explanation of this result.

Exercise 7

Let $x(t)$ be the periodic signal shown in Fig. 2.

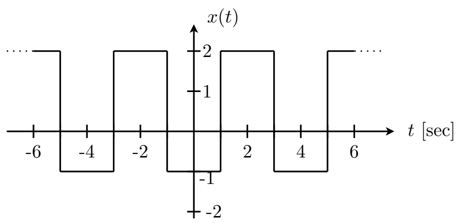

Figure 2: Plot of periodic signal $x(t)$.

Since $x(t)$ is a periodic signal with fundamental period $T_0 = 1/F_0$ we can write it by its Fourier series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$$

a. Consider the signal $y(t)$ shown in Fig. 3, which is related to $x(t)$ by $y(t) = 2x(t) + 3$. This signal is clearly again a periodic signal with the same fundamental period T_0 as $x(t)$ so we can write this signal by its Fourier series expansion:

$$y(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{j2\pi F_0 kt}$$

Express the Fourier series coefficients for this signal, β_k, in terms of the coefficients α_k for $x(t)$.
Hint: This is a simple relationship, and finding it should not require that you compute any of the coefficients explicitly.

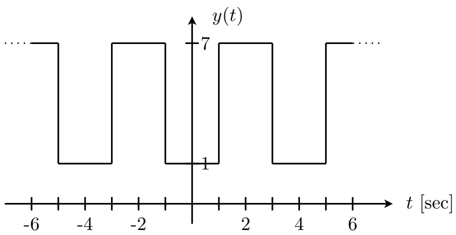

Figure 3: Plot of periodic signal $y(t)$.

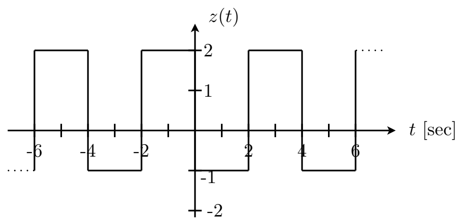

Figure 4: Plot of periodic signal $z(t)$.

b. Consider the periodic signal $z(t)$ shown in Fig. 4, which is related to $x(t)$ by $z(t) = x(t-1)$. This signal $z(t)$ has again the same fundamental period T_0 as $x(t)$ so we can write this signal by its Fourier series expansion:

$$z(t) = \sum_{k=-\infty}^{\infty} \gamma_k \mathrm{e}^{\mathrm{j}2\pi F_0 k t}$$

Express the Fourier series coefficients for this signal, γ_k, in terms of the coefficients α_k for $x(t)$. Again, this is a simple relationship, and finding it should not require that you compute any coefficients explicitly.

Exercise 8

Write the signal $x(t) = \cos^3(100\pi t)$ as a Fourier series, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}\omega_0 k t}$$

with fundamental frequency $\omega_0 = 2\pi F_0$.

Exercise 9

Consider the time-domain plots and frequency spectra shown below, as well as the time-domain formulas and Fourier series coefficients listed below the figures. Together, these eight signal representations (R1-R8) describe four different signals. Each signal is characterized by two of these representations. Link the corresponding signal representations.

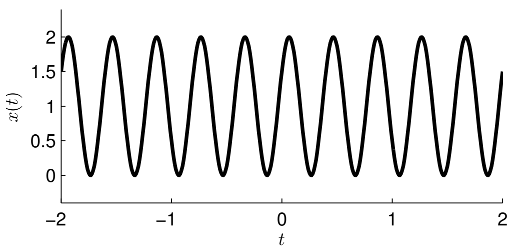

Figure 5: R1: time-domain plot

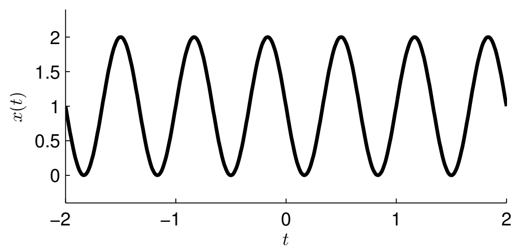

Figure 6: R2: time-domain plot

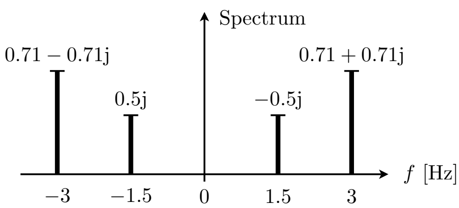

Figure 7: R3: frequency spectrum

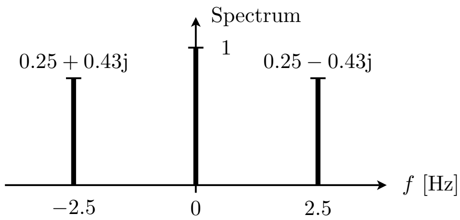

Figure 8: R4: frequency spectrum

- R5: time-domain formula $x(t) = \cos\left(2\pi 1.5t - \frac{\pi}{2}\right) + 2\cos\left(2\pi 3t + \frac{\pi}{4}\right)$
- R6: time-domain formula $x(t) = 2\cos\left(2\pi 1.5t + \frac{\pi}{4}\right) + \cos\left(2\pi 2.5t - \frac{3\pi}{2}\right)$
- R7: Fourier series with $\omega_0 = 3\pi$ and coefficients $\alpha_k = \begin{cases} -j\frac{1}{2} & k=-1 \\ 1 & k=0 \\ j\frac{1}{2} & k=1 \\ 0 & \text{otherwise} \end{cases}$

- R8: Fourier series with $\omega_0 = \pi$ and coefficients $\alpha_k = \begin{cases} -j\frac{1}{2} & k=-5 \\ -\frac{1}{2}\sqrt{2} - j\frac{1}{2}\sqrt{2} & k=-3 \\ \frac{1}{2}\sqrt{2} + j\frac{1}{2}\sqrt{2} & k=3 \\ j\frac{1}{2} & k=5 \\ 0 & \text{otherwise} \end{cases}$

Exercise 10
The signal $x(t)$ is a periodic triangular signal, for which $x(t) = x(t+T_0)$ holds. A complete description of $x(t)$ is given by the following formula for one period of $x(t)$

$$x(t) = \begin{cases} \frac{2t}{T_0} & \text{for } 0 \le t \le \frac{T_0}{2} \\ 2 - \frac{2t}{T_0} & \text{for } \frac{T_0}{2} < t < T_0 \end{cases}$$

Since $x(t)$ is a periodic signal with fundamental period $T_0 = 1/F_0$ we can write it by its Fourier series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}2\pi F_0 k t}.$$

a. Make a sketch of the periodic function $x(t)$ for $|t| \le 2T_0$.

b. Determine the DC coefficient of the Fourier Series, α_0.

c. Use the Fourier analysis integral

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}2\pi F_0 k t}\mathrm{d}t$$

to determine a general formula for the Fourier Series coefficients α_k. Your final result for α_k should depend on k.
Note: You can use the following integral:

$$\int_A^B x\mathrm{e}^{-x}\mathrm{d}x = -(x+1)\mathrm{e}^{-x}|_A^B$$

d. In practice we approximate such a periodic signal with a finite number of harmonics as follows:

$$\hat{x}(t) = \sum_{k=-N}^{N} \alpha_k \mathrm{e}^{\mathrm{j}2\pi F_0 k t}$$

Make a sketch of $\hat{x}(t)$ for $N = 1$ showing that this approximation with one harmonic is a reasonable approximation of $x(t)$.