Exercises

Module Basics Sampling and Reconstruction

Notes:

- Only the answers are available.
- The symbol [P] in the margin of an exercise denotes there is a pencast available.

Exercise 1

[P1] Let $x(t) = 3\sin(20\pi t - \pi/3)$ be the input signal of the ideal C/D convertor as depicted in Fig. (1).

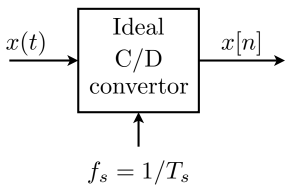

Figure 1: Ideal C/D converter.

In each of the following cases the discrete-time signal samples $x[n]$ are obtained by sampling $x(t)$ at a sample rate f_s [samples/sec] and the resulting signal samples $x[n]$ can be written as

$$x[n] = A\sin(\theta n + \varphi)$$

with $A > 0$, $0 < \theta < \pi$ and $|\varphi| < \pi$. For each case below, determine the values of A, φ and θ. Make a spectral plot of $x[n]$ as a function of the relative frequency θ [-] in the Fundamental Interval ($|\theta| \leq \pi$) and compare this plot with the spectral plot of $x(t)$ as a function of absolute frequency f in [Hz].

a. Let the sampling frequency be $f_{sa} = 24$ [samples/sec].

b. Let the sampling frequency be $f_{sb} = 16$ [samples/sec].

c. Let the sampling frequency be $f_{sc} = 8$ [samples/sec].

Exercise 2

The continuous-time signals $x_1(t)$ and $x_2(t)$ are given as

$$x_1(t) = 2\cos(4\pi t), \quad x_2(t) = 2\cos(6\pi t).$$

a. Carefully plot the signals $x_1(t)$ and $x_2(t)$ in one figure such that at least four periods of each signal are shown.

b. In your plot, identify the points for which $x_1(t)$ and $x_2(t)$ are equal. At what frequency do these points occur (i.e., how many times per second do we have $x_1(t) = x_2(t)$)?

c. Suppose that the signals $x_1(t)$ and $x_2(t)$ are now sampled with exactly this frequency. We thus obtain the sampled signals $x_1[n]$ and $x_2[n]$. Obtain a formula for $x_1[n]$ and $x_2[n]$ to show that these samples are equal.

Exercise 3

A signal $x(t)$ has the two-sided spectrum representation shown in Fig.(2).

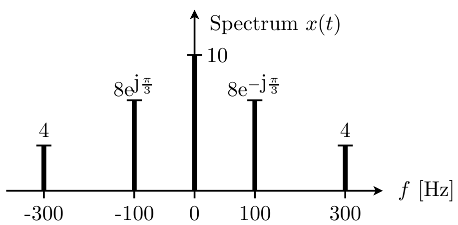

Figure 2: Frequency spectrum of $x(t)$.

The signal $x(t)$ is sampled with sampling frequency $f_s = 1/T_s$ [samples/sec] to obtain the discrete-time signal samples $x[n] = x(t)|_{t=n \cdot T_s}$.

a. Give an equation for $x(t)$ as sum of sinusoids.

b. Give an equation for $x[n]$ and plot the spectrum of $x[n]$ as a function of the relative frequency θ within the Fundamental Interval $-\pi \leq \theta \leq \pi$ when $f_s = 900$ [samples/sec].

c. Do the same as in the previous question, but now for $f_s = 300$ [samples/sec].

Exercise 4
A signal $x(t)$ has the two-sided spectrum representation shown in Fig.(3).

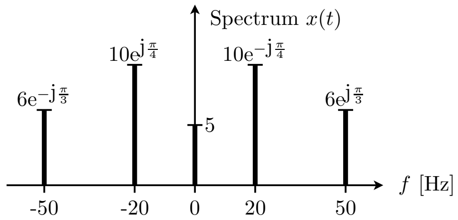

Figure 3: Spectrum of $x(t)$.

a. Write an equation for the signal $x(t)$ and a sum of sinusoids.

b. Suppose that the signal is sampled to produce the sequence $x[n] = x(t)|_{t=nT_s}$, where $1/T_s = f_s = 200$ [Hz]. Sketch the spectrum of the sampled signal as a function of the relative frequency θ in the range: $-2\pi < \theta < 2\pi$ (i.e. show the alias frequencies). Thus make a spectral plot of all the frequency components of $x[n]$ not only within the Fundamental Interval, but also outside this interval, i.e. show also alias frequencies.
Carefully label the complex weights of the frequency components and verify which frequency components are the same.

Exercise 5
A signal $x(t)$ has the two-sided spectrum representation shown in Fig. (4).

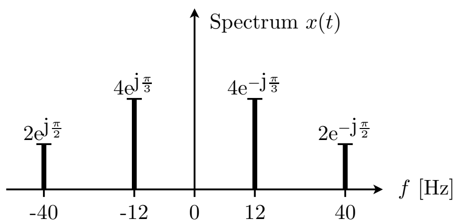

Figure 4: Spectrum of $x(t)$.

a. Is $x(t)$ a real valued signal? If so, explain why?

b. Write an equation for the signal $x(t)$ and a sum of sinusoids.

c. Is $x(t)$ a periodic signal? If so, what is its period?

d. Determine the minimum sampling rate f_s [samples/sec] that can be used to sample $x(t)$ without any aliasing.

Exercise 6
Signal $x(t)$ is represented by the following equation:

$$x(t) = \sum_{k=-2}^{2} (\alpha_k)\, \mathrm{e}^{\mathrm{j}2400\pi k t}$$

with $\alpha_0 = \frac{1}{\sqrt{3}}$, $\alpha_k = \frac{1}{\sqrt{3}+\mathrm{j}(\sqrt{3})^k}$ for $k = 1, 2$ and $\alpha_{-k} = \alpha_k^*$.

a. Is this signal periodic? If so, what is the Fundamental period T_0?

b. Write an equation for the signal $x(t)$ and a sum of sinusoids.

c. Determine the minimum sampling rate f_s [samples/sec] that can be used to sample $x(t)$ without any aliasing.

Exercise 7
[P2] Consider the sampling system shown in Fig. (1) with sampling rate $1/T_s = f_s = 2500$ [samples/sec]. Suppose that the discrete-time signal $x[n]$ is given by the formula

$$x[n] = 10 \cos\left(\frac{2\pi}{5} n - \frac{\pi}{3}\right).$$

Many *different* continuous-time signals could have been the input to the above system. Determine two such input signals with frequency less than 2500 [Hz]; i.e., find a continuous time $x_1(t) = A_1 \cos(2\pi f_1 t + \phi_1)$ and another continuous time signal $x_2(t) = A_2 \cos(2\pi f_2 t + \phi_2)$ such that $x[n] = x_1(t)|_{t=nT_s} = x_2(t)|_{t=nT_s}$.

Exercise 8
Consider the D/C convertor as depicted in Fig. 5.

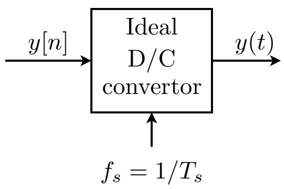

Figure 5: Ideal D/C converter.

The discrete-time signal samples can in general be described with the equation: $y[n] = \sin(\theta_0 n + \frac{\pi}{3})$. The D/C convertor runs at a sample rate of f_s [samples/sec]. The continuous-time signal can in general be described as $y(t) = A \sin(2\pi f_0 t + \phi)$, with $A > 0$, $f_0 > 0$ and $\phi > 0$.

a. Assume $\theta_0 = \frac{\pi}{2}$ and $f_s = 20$ [samples/sec]. Derive a mathematical expression for $y(t)$ and give a plot of the spectra from $y[n]$ and $y(t)$.

b. Assume $\theta_0 = \frac{\pi}{2}$ and $f_s = 32$ [samples/sec]. Derive a mathematical expression for $y(t)$ and give a plot of the spectra from $y[n]$ and $y(t)$.

c. Assume $\theta_0 = \frac{3\pi}{2}$ and $f_s = 20$ [samples/sec]. Derive a mathematical expression for $y(t)$ and give a plot of the spectra from $y[n]$ and $y(t)$.

d. Assume $\theta_0 = \frac{3\pi}{2}$ and $f_s = 32$ [samples/sec]. Derive a mathematical expression for $y(t)$ and give a plot of the spectra from $y[n]$ and $y(t)$.

Exercise 9
Consider the sampling and reconstruction system shown in Fig. (6).

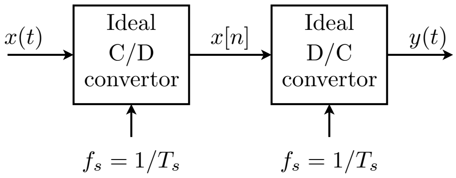

Figure 6: Sampling and reconstruction system.

In this figure, $x[n] = x(t)|_{t=nT_s}$ and $x(t)$ is given by the formula

$$x(t) = 10\cos(10\pi t) + 2\cos\left(40\pi t + \frac{\pi}{3}\right).$$

a. What condition must be satisfied by the sampling rate, $f_s = 1/T_s$, such that $y(t) = x(t)$?

b. What is the output $y(t)$ if the sampling rate is $f_s = 20$ [samples/sec]? Sketch the spectrum of the sampled signal $x[n]$ (as a function of the relative frequency θ for relative frequencies $-2\pi < \theta < 2\pi$, i.e. show also aliased frequencies and label the axis carefully.

Exercise 10
Consider again the sampling and reconstruction system of Fig. (6). The ideal C-to-D converter samples $x(t)$ with a sampling period $T_s = 1/f_s$ [sec] to produce the discrete-time signal samples $x[n]$. The ideal D-to-C converter then forms a continuous-time signal $y(t)$ from the samples $x[n]$. Suppose that $x(t)$ is given by

$$x(t) = (10 + 10\cos(500\pi t - \pi/2)) \cdot \cos(2000\pi t).$$

a. Use Eulers formulas to expand $x(t)$ in terms of complex exponentials and sketch the two-sided spectrum of this signal. Be sure to label important features of the plot. Is this waveform periodic? If so, what is the period?

b. What is the minimum sampling rate f_s that can be used in the above system such that $y(t) = x(t)$?

c. Plot the spectrum of the discrete-time signal samples $x[n]$ for the case when $f_s = 5000$ [Hz] as a function of the relative frequency θ with $|\theta| \leq \pi$.

Exercise 11
Consider again the sampling and reconstruction system of Fig. (6). In all parts below, the sampling rates of the C/D and D/C converters are equal, and the input to the Ideal C/D converter is

$$x(t) = 2\cos\left(100\pi t + \frac{\pi}{2}\right) + \cos(300\pi t).$$

a. If the output of the ideal D/C Converter is

$$y(t) = x(t) = 2\cos\left(100\pi t + \frac{\pi}{2}\right) + \cos(300\pi t),$$

what general statement can you make about the sampling frequency f_s in this case?

b. If the sampling rate is $f_s = 250$ [samples/sec], determine the discrete-time signal samples $x[n]$, and give an expression for $x[n]$ as a sum of cosines. *Make sure that all frequencies in your answer are positive and less than π [radians].* Plot the spectrum of this signal over the range of relative frequencies $-\pi < \theta < \pi$. Label the frequency, amplitude and phase of each spectral component.

c. If the output of the ideal D/C Converter is

$$y(t) = 2\cos\left(100\pi t + \frac{\pi}{2}\right) + 1,$$

determine the value of the sampling frequency f_s.

Exercise 12

[P3] Consider the sampling and reconstruction system shown in Fig. (7).

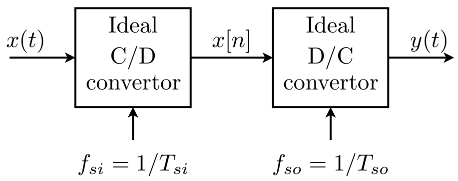

Figure 7: Sampling and reconstruction system with different sampling rates f_{si} and f_{so}.

The input signal $x(t)$ is given as

$$x(t) = \cos\left(400\pi t + \frac{\pi}{4}\right).$$

Both the C-to-D converter and the D-to-C converter can be operated at two different sampling frequencies, which are 400 [Hz] and 800 [Hz]. Explain which sampling frequency you would choose for each converter to make sure that the output signal is given as

$$y(t) = \cos\left(200\pi t + \frac{\pi}{4}\right).$$

Exercise 13

Consider the sampling and reconstruction system of Fig. (7).

a. Suppose that the input $x(t)$ is given by

$$x(t) = 10 + 10\cos(4000\pi t - \pi) + 8\cos(14000\pi t - \frac{3\pi}{4}).$$

Determine the spectrum for $x[n]$ when $f_{si} = 10000$ [samples/sec]. Make a plot for your answer, making sure to label the frequency, amplitude and phase of each spectral component.

b. Using the discrete-time spectrum from part (a), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{so} = f_{si} = 10000$ [Hz].

c. Again using the discrete-time spectrum from part (a), determine the analog frequency components in the output $y(t)$ when the sampling rate of the D-to-C converter is $f_{so} = 20000$ [Hz]. In other words, the sampling rates of the two converters are different.

Exercise 14

Consider again the sampling and reconstruction system of Fig. (6). We can do some interesting things with sampling. One of them is that we can change the period of a periodic waveform. This problem illustrates how this can be done for the specific periodic input signal

$$x(t) = 2\cos(66\pi t) + \cos(198\pi t).$$

In all the following parts, assume that the sampling frequency is $f_s = 30$ [Hz]. Note that this sampling rate does not satisfy the conditions of the Shannon sampling theorem, so aliasing will occur.

a. Plot the spectrum of the periodic continuous-time signal $x(t)$. What is the fundamental frequency of $x(t)$?

b. Determine an expression for the discrete-time signal $x[n]$ as a sum of discrete-time cosine signals. Be sure that all of the relative frequencies are positive and less than π radians. Plot the spectrum of $x[n]$ over the range of relative frequencies $-\pi \leq \theta \leq \pi$.

c. Now the continuous-time output signal $y(t)$ that is created by the ideal D-to-C converter operating with sampling rate $f_s = 30$ [Hz] will also be a sum of cosine signals. Write an expression for $y(t)$ and plot its spectrum. What is the fundamental frequency of $y(t)$?

d. How are the fundamental frequencies of $x(t)$ and $y(t)$ related? Do you think that it would be possible to change the fundamental frequency by a different factor by using a different sampling rate?